Classifying definable linear and partial orders in tame ordered structures.

Janak Ramakrishnan

CMAF, University of Lisbon
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First Question

- The definition (Pillay-Steinhorn): an o-minimal structure is an ordered structure, M, such that every definable subset of M is a union of finitely many points and intervals.
- Let M be an o-minimal group. Let $(P, \prec)$ be a M-definable total linear order. What does P look like?
- The simplest definable linear orders are the lexicographic ones on M^n. We use $<_{\text{lex}}$ to denote the lexicographic order. Obviously, a definable linear order can be a definable subset of such a lexicographic order, or the image of such a subset under a definable injection.
- Theorem: That's it.

First Answer

Theorem A

Let M be an o-minimal field, let $(P, \prec)$ be an M-definable linear order with $n = \dim(P)$. Then there is an embedding g of $(P, \prec)$ in $(M^{n+1}, <_{lex})$. The projection of $g(P)$ to the last coordinate is finite.

Remark

We actually only need M to eliminate imaginaries and possess a definable order-reversing bijection from M to M. Then g maps P to M^{2n+1}, with finite projections to the odd coordinates.

Prior Work

- Steinhorn has unpublished work that implies Theorem A when $\dim(P) = 1$.
- Steinhorn and Onshuus recently showed that a definable linear order could be broken up into finitely many pieces, on each of which Theorem A held.
- However, their result did not say how the order compared elements in different pieces, so the study of definable linear orders could not be reduced to the study of definable subsets of lexicographic orders.
- They also noted that such a result has applications in economics.

One-dimensional interleaving

Example

Let $P = (0,1) \cup (1,2)$, with the order $\prec$ defined to agree with $<$ on $(0,1) \times (0,1)$ and $(1,2) \times (1,2)$, and defined as $a \prec b$ iff $a \leq b - 1$ on $(0,1) \times (1,2)$.

$$.25 \prec .5 \prec 1.5 \prec .75 \prec 1.8 \prec 1.9$$

The embedding:

Send $a \in (0,1)$ to $\langle a, 0 \rangle$. Send $b \in (1,2)$ to $\langle b-1, 1 \rangle$.

One dimension

The key result needed in the proof of the one-dimensional case, also needed in the n-dimensional case, is the following:

Fact B

Let $\prec$ be any definable linear order on Γ, a definable curve. Then there is a finite partition of Γ such that on each set in the partition, the induced order coming from the curve parametrization is just $\prec$ or $\succ$.

Using Fact B, we can take any one-dimensional order and break it up into pieces, on each of which we have the standard order. The only remaining task is to classify the ways in which these pieces fit together, which is messy but routine.

Induction: an equivalence relation of closeness

Definition

For $x, y \in P$, let xEy if the $\prec$-interval bounded by x and y has dimension $< n$.

E is a $\prec$-convex equivalence relation on P.

Lemma

No E-class has dimension n.

Proof.

If not, fix such an E-class, A, and take $b \prec c$ two "extreme" elements of A^* in an elementary extension M^*. Then $b \prec A \prec c$, so $\dim((b,c)_\prec) = n$, contradicting bEc. □

Induction: full-dimension points

Definition

Let H be the set $\{x : \forall y < x(\neg yEx)\}$.

H consists of all points with "full dimension" below. We will see that $\dim(H) < \dim(P)$ if $\dim(P) > 1$.

Note that in the classic example of lexicographic order, the set H is empty, since points in the same vertical fiber are in the same E-class.

Induction or else

Lemma

If $\dim(H) < n$*, then Theorem A follows.*

- The premise implies that P/E has dimension $< n$: otherwise there is an n-dimensional set of finite E-classes. The least element of each such class is in H, giving $\dim(H) = n$.
- By induction, P/E definably embeds in a lexicographic order. Also by induction, for each $x \in P$, the class $[x]_E$ definably embeds in a lexicographic order.
- With some careful stitching together while keeping track of dimensions, the theorem is proved.

Or else

Lemma

If $\dim(H) = n$*, then* $n \leq 1$*.*

- The set $H' = \{x : [x]_E = \{x\}\}$ has dimension n because H does.
- We follow a technique of Hasson and Onshuus, and pick a definable curve $\Gamma \subseteq H'$ on which $<$ and $\prec$ agree.
- Each element of P defines a "cut" in the $\prec$-order on Γ, and so we have the Γ-definable equivalence relation of lying in the same $\prec$-cut. O-minimality ensures that this equivalence relation is definable.
- By a standard fiber argument, we can find an $(n-1)$-dimensional equivalence class.
- This equivalence class is convex and contains a point of Γ, contradicting that this point is in H' unless the equivalence class consists of a single point, i.e., $n - 1 = 0$.

Second Question (joint with C. Steinhorn)

J. Truss asked whether any definable partial order in an o-minimal structure could be definably extended to a linear order. Here, we will positively answer a generalization of this question, by describing several classes of ordered structures that definably extend their definable partial orders. These structures can be thought of as possessing a "definable" order extension principle – in these structures, the "order extension principle" of ZFC holds definably. Formally:

Definition

Let M be a structure. Say that M *has the order extension principle* (*has OE*) if for any M-definable partial order $(P, \prec)$, there is an M-definable linear order $\prec'$ that totally orders P and such that $x \prec y \Rightarrow x \prec' y$.

Examples of structures with OE

A structure M has OE if it definably extends any partial order to a total one.

1. All well-ordered structures.
2. All (weakly) o-minimal structures (every definable 1-dimensional set is uniformly a finite union of points and convex sets).
3. Definably complete quasi-o-minimal structures.

The case of a partial order on M when M is well-ordered is quite easy, and appears in a paper of Felgner and Truss. The method they use is actually a specialization to the 1-dimensional case of our method.

The key easy step

All this work hinges on an easy observation: that any partial order can be represented by a definable family of sets.

Definition

Let $(P, \prec)$ be a partial order. Let $L(x) = \{y \in P : y \prec x\}$ for $x \in P$ – the "lower cone" of x. Let $\mathcal{V}_P = \{L(x) : x \in P\}$. The partial order $\prec_V$ on P is defined by $x \prec_V y$ if $L(x) \subsetneq L(y)$.

Note that if $x \prec y$, then by transitivity and the fact that $x \in L(y) \setminus L(x)$, we have $x \prec_V y$, so $\prec_V$ is a partial order on P extending $\prec$.

To show M has OE, we need to show that if P is the parameter set of a definable family of sets $\mathcal{V}$, then P can be linearly ordered, compatible with the subset order.

Theorem

Let M be a well-ordered structure. Then M has OE. [Already implied by Felgner and Truss.]

Proof.

Let P be the parameter set for $\mathcal{V} = \{V(x) : x \in P\}$, a definable family of sets in M^n for some $n \geq 0$. We consider the case $n = 1$. The general case is similar.

For $x, y \in P$, let $B(x, y) = V(x) \triangle V(y)$. Since M is well-ordered, there is a least element of $B(x, y)$. Then for $x, y \in P$, let $x \prec y$ if $t \in V(y)$ (so $t \notin V(x)$).

If x and y are still unordered, then $V(x) = V(y)$. Order x and y lexicographically. $\square$

The question, transformed

Any partial ordered set parametrizes a family of sets, and the partial order of set inclusion induces an extension, $\prec_V$, of the original partial order.

We have shown that if we can linearly extend partial orders coming from definable families, we can linearly extend all partial orders. Since we are now dealing with definable families, we can consider fibers.

Given $\mathcal{V} = \{V(x) : x \in P\}$, a definable family of sets in M^n, we have a partial order $\prec_V$ induced on P.

To compare elements $x, y \in P$, look at $V(x) \triangle V(y)$ and go by the least element of this set.

For higher dimensions, we use the fact that for any $t \in M$, we can consider the family $\mathcal{V}_t = \{V(x)_t : x \in P\}$. This induces a partial order $\prec_t$ on P. The collection $\mathcal{V}_t$ is a family of $(n-1)$-dimensional sets and so, by induction, we may suppose that each $\prec_t$ is actually a linear order on P.

Instead of letting $B(x, y) = V(x) \triangle V(y)$, we set $B(x, y) = \{t : V(x)_t \neq V(y)_t\}$. Then we let $x \prec y$ if $x \prec_t y$ for t the least element of $B(x, y)$.

The general case

The previous proof gives the principle for subsequent proofs: if there is some consistent way to pick out a particular part of $B(x, y)$, for which each $\prec_t$ gives the same answer about x and y, then we can use that answer to order x and y.

Theorem C

Let M be an ordered structure such that, for any definable $A, C \subseteq M$, there is some initial segment of C either contained in or disjoint from A. Then M has OE.

Claim: Any ordered structure with tame initial segment behavior of definable sets has OE.

Proof.

- As before, we restrict to the 1-dimensional case for simplicity.
- The proof proceeds as in the well-ordered case until we have $B(x, y) = V(x) \triangle V(y)$.
- Consider the definable set $\{t : t \in V(y) \setminus V(x)\}$. By hypothesis, this set either contains or is disjoint from an initial segment of $B(x, y)$.
- If it contains an initial segment of $B(x, y)$, then set $x \prec y$. Otherwise, let $y \prec x$.
- It is then routine to verify that this yields a nearly-total order, which is completed lexicographically.

□

Consequences of Theorem C

Theorem C

If M is an ordered structure such that for any definable $A, C \subseteq M$, A contains or is disjoint from an initial segment of C, then M has OE.

Theorem C immediately implies our results on well-ordered, o-minimal, and weakly o-minimal structures.

Note that while the hypothesis on M in Theorem C is first-order, the properties of being well-ordered or weakly o-minimal are not first-order. Thus, if *some* model of the theory of M is weakly o-minimal or well-ordered, then M satisfies the requisite hypothesis.

Extending the proof

As referred to before, if there is some consistent way to pick out a particular part of $B(x, y)$, for which each $\prec_t$ gives the same answer about x and y, then we can use that answer to order x and y.

We thus describe a class of structures for which a more intricate model-theoretic argument works.

Recall that an ordered structure M is *definably complete* if any definable $A \subset M$ has a greatest lower bound $a \in M \cup \{-\infty\}$.

Recall that a structure is *ω-saturated* if any type over finitely many parameters is realized in the structure itself.

Confusing property

Consider the property (‡) for an ordered structure M: For any element $d \in M$ and any complete 1-type $p \in S_1(d)$, any M-definable set $A \subseteq M$ is either disjoint from or contains an initial segment of the set of realizations of p.

This is in some sense a natural generalization of the previous property we looked at. Here, instead of considering the initial segment of a single definable set, we might have to take the intersection of infinitely many of them before looking at the initial segment – the type replaces the set C that we used previously, and the set A remains the same.

If A contains an initial segment of p's realizations, then we say that p *believes* A.

Definably complete: lower boundaries of definable sets exist in the structure; ω-saturated: types over finite sets are realized; (‡): any definable set contains or is disjoint from an initial segment of a type.

Theorem D

Let M be a definably complete, ω-saturated ordered structure with (‡). Then M has OE.

The proof proceeds as before, but the definition of the order in terms of $B(x, y)$ will be considerably more complicated, due to multiple applications of compactness.

Claim: Let M be a definably complete, ω-saturated ordered structure with any definable set containing or disjoint from an initial segment of certain 1-types. Then M has OE.

Let $x, y \in P$. As before, we restrict to the one-dimensional case. We want to look at $V(y) \setminus V(x)$ on an "initial segment" of $B(x, y)$. However, $V(y) \setminus V(x)$ may not behave nicely on an initial segment of $B(x, y)$, so we must consider it on types near the lower boundary of $B(x, y)$.

Let $d = d(x, y)$ be the lower boundary of $B(x, y)$. Let $\mathcal{Q}_d$ be the set of all types over d that have realizations coinitial above d.

- Our first goal is to "finitize" property (‡). For an arbitrary definable set A, we only know that there is constant initial segment behavior on a type, not on an arbitrary definable set C.
- However, if we fix a family of definable sets, say given by $\psi(t, a)$, then using compactness we can find a definable set $\varphi(t)$ in p such that $\psi(t, a)$ will have constant initial segment behavior on φ.
- This puts us in a situation very close in spirit to Theorem C's.

Claim: Let M be a definably complete, ω-saturated ordered structure with any definable set containing or disjoint from an initial segment of any type. Then M has OE. $d(x, y)$: lower boundary of $B(x, y) = V(x) \triangle V(y)$; The set $\mathcal{Q}_d$ has all types with realizations coinitial above d.

- Standard compactness shows us that since the sets $V(y) \setminus V(x)$ and $V(x) \setminus V(y)$ each contain or are disjoint from an initial segment of p (by ‡), they must contain or be disjoint from an initial segment of some formula, φ_p, in p. Repeating compactness, we may suppose that the same formula φ_p has this property for all $x, y \in P$ with the same value for $d(x, y)$.
- For any $x, y \in P$, it is fairly routine to show that there is some type in $\mathcal{Q}_d$ that believes $B(x, y)$ – we can simply construct it, since $B(x, y)$ is coinitial above d.
- Compactness again, applied to the formulas φ_p for each $p \in \mathcal{Q}_d$ that believes $B(x, y)$, shows that there are $\varphi_1(t, d), \ldots, \varphi_N(t, d)$ such that every $x, y \in P$ with $d(x, y) = d$ has $B(x, y)$ believed by some φ_i.
- One more application of compactness gives the φ_i's uniformly as we vary d.

Claim: Let M be a definably complete, ω-saturated ordered structure with any definable set containing or disjoint from an initial segment of certain 1-types. Then M has OE.

- We have $\varphi_1(t, z), \ldots, \varphi_N(t, z)$ with the property that for any $x, y \in P$, for some i we have $B(x, y)$ believed by $\varphi_i(t, d(x, y))$.
- Moreover, the sets $V(y) \setminus V(x)$ and $V(x) \setminus V(y)$ each contain or are disjoint from some initial segment of each $\varphi_j(M, d(x, y))$.

Thus, we define $x \prec y$ to hold if, on the least i such that $B(x, y)$ is believed by $\varphi_i(t, d(x, y))$, we have $V(y) \setminus V(x)$ believed by $\varphi_i(t, d(x, y))$.
Verification that this is a partial order extending the original is routine, since in some sense it is a "lexicographic" order based on the behavior on each φ_i.

Consequences

This much more technical result unfortunately has limited application. It most directly deals with quasi-o-minimal structures: ordered structures in which every definable set is (uniformly) a finite Boolean combination of points, intervals, and $\emptyset$-definable sets. Even those, though, must also be definably complete.
The definable completeness requirement can be relaxed if M eliminates imaginaries, so by passing to M^{eq}. However, it is possible that such a change would break property (‡) – in doing this, we add new elements that are in some sense "less powerful" than other elements were, so types over these new elements might not be fine enough to have initially constant behavior with respect to a definable set.

Finite orders

Theorem C may have some application in the case of finite orders. While algorithms are known for extending partial orders that are linear in the number of elements plus the number of relations, this may be quite slow for sufficiently large partial orders.
Theorem C gives a uniform way to decide how to order two elements, given an (unrelated) total order on the elements. That is, given $a, b \in P$, the only information necessary is $L(a)$ and $L(b)$, the lower cones. Thus, if the partial order is furnished with the lower cones given, then an algorithm can decide if $a \prec b$.
Notably, the algorithm does not have to order the entire partial order, or repeatedly take the transitive closure of an order as it decides more and more. The total order gives a way to order elements pairwise without fear of inconsistency.

Partial orders are hard

Definably extending a partial order to a linear order turns out to be possible in many cases. In the o-minimal case, in particular, this implies that any partial order can be considered to be a reduct of a suborder of a lexicographic order.
However, we still know very little about partial orders in o-minimal structures. Even in one dimension, we do not have a complete classification of partial orders, and the outlook is not very promising.

Generalizing

This technique may have been pushed as far as it can be. But the general question of whether an ordered structure M has OE is still open, as far as I know. In particular, I do not have an example of an ordered structure M that does not have OE, although it seems likely that there is one, since in general the total and partial orders need have nothing in common. However, looking at the case of well-ordered structures, the condition on the structure is just about the order, with no assumption made about the interaction between the partial and total orders. Thus, it is not clear that there needs to be anything in common between the partial and total orders so that the total order can definably linearly extend the partial order. Particular cases of interest are ordered dp-minimal and ordered NIP structures – ones in which the total order might have more control over the definable sets.