1 Introduction

My research is in o-minimality, a branch of model theory with connections to real analysis and algebraic geometry. An o-minimal structure is a totally ordered structure in which every one-dimensional definable set is a finite union of points and intervals [vdD98]. The reals as a field form an o-minimal structure [vdD98]. It remains o-minimal even when the exponential function is added [Wil96]. An o-minimal field (that is, an o-minimal structure expanding a field) is necessarily real closed. The framework of o-minimality gives a general context for the study of real algebra [Kra00, KF06], topology [EW08], geometry [BL07, Ber07], and analysis [Iof07, Spe99]. It has yielded new results in the representation theory of Lie groups [SV00] and real analysis [RSS07, vdDS00] and is interesting in its own right as a branch of model theory. I completed my Ph.D. under Professor Thomas Scanlon, at the University of California, Berkeley, in December 2008.

2 Extending Functions on Curves

The bulk of my dissertation [Ram08] (now submitted as [Ram09a]) was devoted to proving the following theorem. It answered a question of P. Speissegger, about extending a bounded function to a closed set containing a (not necessarily definable) curve. It arose for him in the study of differential equations, specifically trying to prove an analogue to a theorem of Malgrange [Mal74] in which the existence of a formal solution to an ordinary differential equation in a space of generalized series implies that there is actually a C^∞ function that is a solution, and that has asymptotic expansion this formal solution.

Theorem 2.1. *Let M be an o-minimal field. Let γ be a non-oscillatory curve in M^n (cartesian product of M) with one endpoint 0. Then the following two statements are equivalent:*

1. *For every M-definable function, f, bounded on some initial segment of γ, there is an M-definable set, C, containing an initial segment of γ such that f is continuous on C and extends continuously to $\operatorname{cl}(C)$.*
2. *Let $\bar{c} = \langle c_1, \ldots, c_n \rangle$ be any "infinitesimal" point of γ, reordered to be decreasing (the required reordering is independent of choice of $\bar{c}$). The type of c_i over $M(\bar{c}_{<i})$ is not near scale or in scale on M, for $i = 1, \ldots, n$.*

M-definable here means definable using a first-order formula in the language of M. Thus, if M is just $\mathbb{R}$ with field structure, then e^x is not an M-definable function, but $x^{3/2}$ is. Scale and decreasing are technical but easy concepts defined in Definitions 2.3 and 2.4.

Speissegger's original hope was that Statement 1 was true for any non-oscillatory γ. However, I discovered the following counterexample.

Example 2.2. *Let M be the reals with the structure of an ordered field, let $\gamma(t) = \langle t, -t/\log t \rangle$, and let $f(x, y) = y/x$. Then there is no semialgebraic set (defined using polynomial equalities and inequalities) containing γ on which f is continuous, and onto whose closure f extends continuously. In particular, f cannot be extended continuously to the origin on any semialgebraic set containing γ.*

This example and one other closely related one turn out to be the prototypes for all instances of failure. However, isolating the precise property that causes failure is non-trivial. The first step is to associate each curve with a certain infinitesimal tuple $\gamma(\epsilon)$ for ϵ an infinitesimal. This makes the curve amenable to the techniques of model theory. We can consider $\gamma(\epsilon)$ as a *type* – the set of all formulas with parameters from M satisfied by $\gamma(\epsilon)$. In fact, the question can be rephrased in the language of model theory, with no reference to curves.

Working on this led me to define the following two concepts. The first is derived from [MS94], although in a somewhat modified form.

Definition 2.3. *Let $M \subseteq N$ be ordered structures. Let $c \notin N$ be not infinitesimally close to any element of N. We say that the type of c over N is* in scale on M *if, for some M-definable function, $f(x, y)$, with x a tuple and y a singleton, and some tuple $b \in N$, $f(b, M)$ is cofinal and coinitial at c in N. Say the type is* near scale on M *if there is a function and tuple, as before, such that $f(b, M)$ is cofinal (or coinitial) at c in N. Say* $\mathrm{tp}(c/B)$ *is* out of scale on A *otherwise.*

Marker and Steinhorn needed a definition like this because they were examining definable n-tuples. To see why we need it, consider Example 2.2 and $f(\epsilon, -)$, for ϵ an infinitesimal. As the argument of $f(\epsilon, -)$ approaches $-\epsilon/\log\epsilon$ from above, $f(\epsilon, -)$ takes on all positive real values, and there is no $M\epsilon$-definable point below which $f(\epsilon, -)$ stops taking positive real values. Equivalently, the inverse of $f(\epsilon, -)$ witnesses that the type of $-\epsilon/\log\epsilon$ over $M(\epsilon)$ is near scale on M, and that equivalence is why scale is useful. However, while scale is a crucial component to the solution, it is not a particularly stable property – for example, it can change under reordering of coordinates. For that reason, I introduced the following notion.

Definition 2.4. *Let M be a structure, and $\bar{a} = \langle a_1, \ldots, a_k \rangle$ be a sequence, with $a_i \notin M$ for $i \leq k$. Say that $\bar{a}$ is* decreasing *if, for each $i < j \leq k$, $(0, b) \cap \mathrm{dcl}(M\bar{a}_{<i}a_j)$ is non-empty for all $b \in \mathrm{dcl}(M\bar{a}_{\leq i})$.*

This notion of decreasing captures precisely that earlier elements are never infinitesimal over later ones. It is not hard to show, as well, that every n-tuple can be reordered to be decreasing. With this in hand, Theorem 2.1 can finally be understood in full.

A few words of explanation of Theorem 2.1: Example 2.2 was near scale, and shows the essential obstacle there. Since in scale is basically near scale on both sides, it is easy to see why failure occurs there as well. The theorem's proof, while technically delicate due to the multiple dimensions, follows the spirit of the above discussion. A key point is that a decreasing type can be contained in a definable set in which any curve with one coordinate going to a limit point must have all later coordinates going to limit points as well, and going to those limit points at least as "fast." This allows us to choose boundary functions for our cell that come together quickly enough to ensure that the function f continuously extends to all boundary points.

3 Pure model theory

The above result led to several new avenues of research in pure model theory. First, while the above concept of "scale" is due to Marker and Steinhorn [MS94], their definition is actually somewhat looser, which makes their characterization of definable types in o-minimal theories not as tight as it could be. I refined their original theorem using my techniques to obtain the following:

Theorem 3.1. *Let* p *be an* n*-type over* M*, and let* $\bar{c} = \langle c_1, \ldots, c_n \rangle \models p$*. Then* p *is definable iff for* $i \leq n$*,* $\mathrm{tp}(c_i/M\bar{c}_{<i})$ *is definable over* $M\bar{c}_{<i}$*, or near scale or out of scale on* M*.*

Scale permits a very fine-grained distinction among types in o-minimal theories. Marker [Mar86] divided o-minimal types into three orthogonal categories, – principal, and two kinds of non-principal types, where principal types are the definable 1-types. With scale, we can further subdivide so that we have five categories of types, all non-interdefinable.

As well, while decreasing types were useful in the above result as a way to control later variables of a type in terms of earlier ones, they can also be studied in their own right. That every type of finite length can be reordered so as to be decreasing is not difficult, but as well, every type of finite length is interdefinable with a decreasing type with the property that it contains no in scale or near scale elements in the sequence.

4 Other results

Besides the work outlined above, I have completed two other papers: "Uniform bounds on growth in o-minimal structures" [Ram09c], and "Maximal small extensions of o-minimal structures" [Ram09b] (which will appear in Mathematical Logic Quarterly). In the first [Ram09c], I show that, given an o-minimal structure M and $f(x,t)$ an M-definable $n+1$-ary function, there is an M-definable unary function g with $f(a,t) \leq g(t)$ for any $a \in M^n$, for sufficiently large t. A result in a paper of Miller and Professor van den Dries ([vdDM96]) implies this when M is an o-minimal field, but it was unknown in several other cases.

My second paper [Ram09b] deals with a general question of Kueker, who asked, for a structure M, is there a "Hanf number" λ such that, if M has a small extension (one realizing no new types over finite subsets of M) of cardinality $> \lambda$, M has small extensions of every cardinality? This question was answered by Hrushovski and Shelah for superstable theories [HS91]. Here, I show the tightness of the theoretical maximal bound for o-minimal structures. Previously, all known examples of models that had small extensions also had unboundedly large (in cardinality) small extensions. Partly this is because any model with only boundedly large small extensions must be non-Archimedean, and in fact, must have infinitely many "levels" at which it is non-Archimedean. Such models had not been constructed before my work.

5 Future research plans

I am currently collaborating on several projects. One concerns dense pairs of o-minimal structures, such as the real closure of the rationals embedded in the reals. I am considering a generalization that includes non-o-minimal pairs of structures, and analyzing what algebraic closure is in this setting. Another project concerns structures such as valued fields – the p-adics and algebraically closed valued fields are two prominent examples that I am working with. The goal is to understand how (model-theoretic) complexity of the value group and residue field governs the complexity of the full field, especially when the field admits a Denef cellular decomposition. A third project is related to Hardy fields [Ros83]. Given an o-minimal field, R, the germs of R-definable functions near ∞ form an ordered differential field, $\mathcal{H}(R)$. When R has underlying set $\mathbb{R}$, then $\mathcal{H}(R)$ is a Hardy field. In general, $\mathcal{H}(R)$ will be an H-field [AvdD02, AvdD05]. Little is known about what structure $\mathcal{H}(R)$ can have. Recently, Lipshitz and

Robinson [LR06] constructed an o-minimal structure R with $\mathcal{H}(R)$ containing all formal power series over $\mathbb{R}$. I am trying to extend their construction to enlarge $\mathcal{H}(R)$. I may be able to construct a structure with $\mathcal{H}(R)$ containing Écalle's field of transseries [Éca92], or even the full field of LE-series [AvdD05, vdDMM01].

Recently I have become interested in a framework developed in [Mař09b, Mař09a, MvdD09] – that of an o-minimal field with a distinguished convex subring, especially when that subring is the convex hull of the reals. The convex subring can be regarded as a valuation ring, leading to the residue field of the valuation. This residue field will have extra induced structure coming from the original o-minimal field, assuming that the field also had additional structure. Maříková and van den Dries have obtained impressive results in this setting, but there are also many questions to be answered. In particular, there is an intriguing connection between such fields with convex subrings and one of the oldest problems in model theory – the decidability of the real field with exponentiation. This would be a major result in decidability theory and model theory, with consequences in transcendental number theory. For instance, there is no known way to test if, given two real algebraic numbers a, b, if $\exp(\exp(a)) = b$. The decidability of the real exponential field would allow one to test this equality for any given a, b. A question that links these fields with convex subrings and the real exponential field is:

Question 5.1. *[Mař09a] Given an ω-saturated o-minimal field, R, and V the convex hull in R of the reals, let k_{ind} be the induced structure on $\mathbb{R}$. Is k_{ind} elementarily equivalent to a reduct of an o-minimal expansion of an elementary extension of R? That is, is the structure induced on $\mathbb{R}$ compatible with an o-minimal expansion of R, possibly after enlarging R?*

This question goes to the heart of how the reals can sit inside a larger o-minimal structure. A positive answer when R has an exponential function would imply the decidability of the real exponential field. This question is modified from the original question asked by van den Dries, which I showed was not always true – I found a counterexample, as well as linked a recent result of T. Foster to another counterexample. The above question has a technical modification that removes these counterexamples, while still implying decidability of the real exponential field.

The concept of scale that I have defined is connected generally to convex subrings. I am working both on adapting my results to this context, and in researching new methods in o-minimality to aid in this project. A key aspect of this question is describing the Dedekind cut (or type) in R that is less than every positive real number, but greater than all "infinitesimal" elements of R. This type is closely related to those that I studied in my dissertation. A full understanding of the behavior of definable functions as they approach the boundary of this type may answer the question fully. In fact, much of the work done so far involves describing the behavior of n-ary functions as they approach this type in terms of unary functions. A crucial part of the work in my thesis involved just this – the reduction of n-ary to unary functions on certain types. Notably, that is the reduction I achieved to prove Theorem 3.1. While work in such areas may seem esoteric, the consequences of a positive answer to Question 5.1 show that it can have far-reaching effects in many parts of mathematics.

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